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Analysis of Bounded Linear Cellular Automata Based on a Method of Image Charges

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The temporal evolution of a class of one-dimensional cellular automata (CA) implementable in VLSI, is a problem of considerable complexity in the presence of the natural (null) boundary conditions. This paper provides an analytical solution to their evolution, based upon a method of image charges borrowed from electrostatics. Reductions in computational effort by factors of $O(L^2)$ for a single site value, or $O(L)$ for the entire configuration of the CA, as compared to direct simulation, are obtainable by the present method of image charges. These results are expected to provide a basis for CA applications as highly parallel computational structures to be incorporated as functional blocks is novel VLSI architectures.

INTRODUCTION

The concept of an array of identical processors arranged on a lattice in one, two or three dimensions was originally proposed by Von Neumann in the 1940s, who named these structures cellular automata [1]. As an alternative to conventional sequential processors, they are capable of highly parallel computation. Although the principles of cellular automata (CA) as discrete systems capable of self-organization, replication and simulation of biological systems were established earlier [1], it is only recently that VLSI integration levels have reached the point where CA structures of interesting complexity levels may be implemented. Systolic arrays may be regarded as CA of relatively large "grain size" [2, 3].

Wolfram [4] has, among others [5], recently explored the properties of CA with the simplest possible processing nodes, which store only one or two bits of information, and involve primary (two-input, one output) logic operations on their immediate neighbour nodes only, in a one-dimensional configuration. In some cases, a 3-input operation is performed by including as input to a node its own state on the previous clock cycle. Wolfram's analysis has included descriptions of the statistical behaviour of large CA arrays, and has related the behaviour of various

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classes of CA (dependent on the logic operations and number of states) to formal language theory of automata, and to dynamical systems.

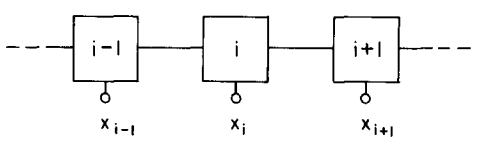
The algebraic properties of a simple class of CA have been studied by Martin *et al.* [6] for the case of periodic boundary conditions. In the present paper we provide some additional mathematical tools which help to simplify the analysis of the case of null boundary conditions.

Null Bounded CA: The Method of Image Charges

In this paper, we discuss the case of one-dimensional cellular automata, in which each processor has 1 bit of memory (i.e., 2 states), and performs the identical logical operation of mod 2 addition (exclusive OR) on the previous state of its left and right neighbours, synchronously on each time step (Fig. 1). This CA corresponds to "Rule 90" according to the classification scheme of Wolfram [4]. These CA are special in that the logic rules are linear; the results of evolution of a global configuration from an initial state in which many cells were "1" may be reconstructed by superposing (mod 2) the results due to each initial cell in turn. They are, however, expected to provide useful models for the understanding of more complex, nonlinear, CA systems [4–6].

An important consideration arises at the boundaries of the array. In a VLSI implementation, we would prefer to hold the end inputs at a constant value, either logic 0 or logic 1. In this paper we assume null boundaries; the "outside" cells at the end of the chain are consistently in the 0 state. Constructions analogous to that presented below may be derived for "1" boundaries. Examples of CA evolution without boundaries, with fixed length and null boundaries, and with periodic boundaries are shown in Fig. 2. These representations are similar to those of Wolfram [4]; the presence of a dot (·) indicates a logic 1, a blank a logic 0. Successive generations appear vertically in the figure.

We have recently demonstrated that one-dimensional CA of this type may be applied as modulo arithmetic units in VLSI, as a consequence of their group theoretic properties [7]. These applications depend upon being able to express, on the basis of initial configurations of global states (a given horizontal line in Fig. 2),



$$x_i\,[T] = x_{i-1}\,[T-1] \oplus x_{i-1}\,[T-1]$$

Fig. 1. One-dimensional cellular automaton (CA) in which each cell updates its binary state on each time step by performing the exclusive OR (mod 2 addition) operation on its two immediate neighbours.

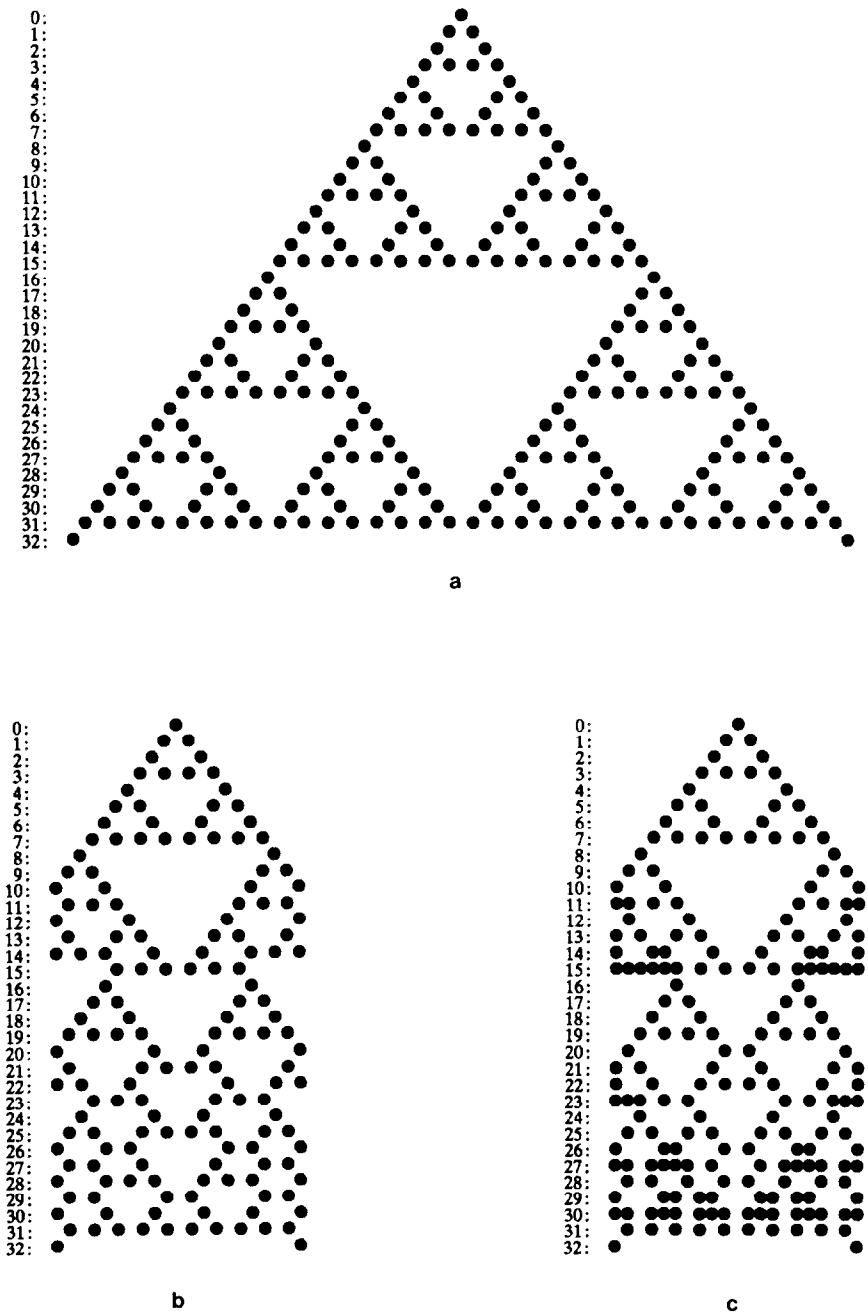


FIG. 2. Evolution of CA configurations according to the logical rule of Fig. 1 with (a) no boundaries, (b) null boundaries, and (c) periodic boundary conditions. Successive time steps are shown vertically down the figure following an initial configuration with a single nonzero site.

the value of any future configuration after an arbitrary number of clock cycles, or time steps.

It turns out that this prediction of future configurations is fairly straightforward for CA of infinite length, in which the boundaries are not encountered by the outgoing "wave" of nonzero sites, as in Fig. 2. Prediction becomes much more complicated with a finite length. In a VLSI system, we would wish to be allowed to "load" the entire register, so that boundaries would be encountered in the first few clock cycles. On the other hand, for efficient operation in the sense of low computation time, we prefer algorithms requiring very few clock cycles.

Figure 2 shows the evolution of the CA configurations from an initial "seed" of a single nonzero site. If the initial configuration were two nonzero sites separated by three cells, two cones such as that of Fig. 2 would be superimposed with a 3-cell horizontal displacement; where two nonzero sites lay on top of one another the result would be zero (sum mod 2). With multiple nonzero seeds, we have multiple horizontally shifted cones, and we count the number of superposed dots; if even, we have 0, if odd a 1 at that site for any given (horizontal) position in the array, and any time step (vertical position).

As discussed by Wolfram [4], the pattern of Fig. 2 for the single nonzero site is a fractal, actually a Sierpinski gasket [8], of fractal dimension $D = \log_2 3$. An alternative representation of the nonzero sites may be arrived at by taking the mod 2 value of the coefficients in Pascals triangle (Fig. 3).

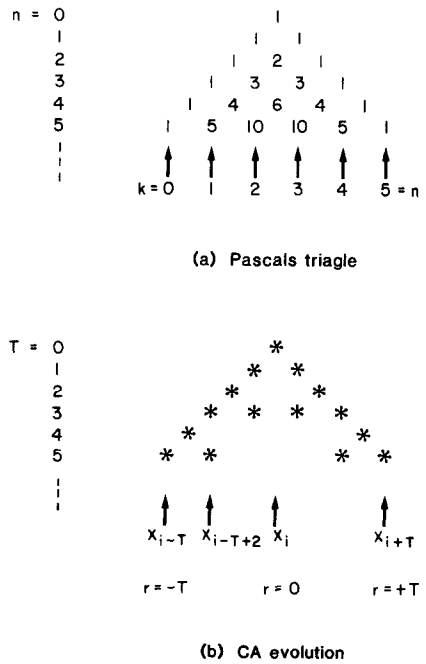


FIG. 3. (a) Coefficients of Pascals triangle, and (b) their mod 2 equivalent shown by an asterisk.

For an unbounded CA in which multiple nonzero sites exist for $T=0$, we can write the value of an arbitrary site of position j at time step T as

$$X_j(T)=\sum_i x_i(0)\binom{T}{(|j-i|+T)/2} \mod 2, \tag{1}$$

where the sum is over the nonzero sites $X_i(0)$ and it is understood that only integer values of $(|j-i|+T)/2$ are included [4]. The influence of the boundaries is readily observed by comparing Fig. 2a and b. The portion of the configuration in Fig. 2a which lies between 0 and L in Fig. 2b is dramatically changed once the "light cone" has reached the boundary.[1] For a nonzero seed placed close to one end, this happens after only a few clock cycles.

Fortunately, the linearity of the CA rule which provided the justification for the superposition embodied in Eq. (1) also rescues us from the boundary complication. In the analysis which follows we have taken our cue from the standard approach to electrostatic calculations for charges in the neighbourhood of perfectly conducting solids. An "image charge" is induced inside the conductor at an equal distance away from the surface as the original charge was on the opposite side. Thereafter the calculation proceeds as if the two charges were present in a vacuum. It only works outside the conductor of course, which in our case is the region of the (bounded) CA.

In Fig. 4, we show the effect of the boundaries upon the evolution from a single nonzero seed, together with the evolution of an unbounded CA from three nonzero seeds. Over the region 0 to L (where $L=8$ in Fig. 4) for the clock cycles from 0 to 15 the results are identical. The additional two seeds in Fig. 4b are the "image charges" which were placed outside the original CA at equal distances from the boundary sites (those just outside the ends of Fig. 4a) as was the original nonzero site. The superposed cones from the original seed and its image combine to maintain the value of these "virtual" boundary sites at a zero value since these sites are equidistant from the two and so either both cones give 0 or both 1, either way this is 0 (mod 2).

A problem eventually arises of course with sufficient time steps because the two image charges introduced produce two additional image charges when reflected about the opposite boundary. That is, the image charge introduced to the left of the array demands an additional image charge off to the right of the array. This second-order image is of course displaced much further away and its effect is not felt until a much later time (of order L time steps). This is analogous to the existence of successive images in a complete electrostatic calculation [9].

[1] The light cone is the advancing front of the nonzero sites; it is so called because the maximum velocity of the front is one site per time step—the "speed of light" in this universe.

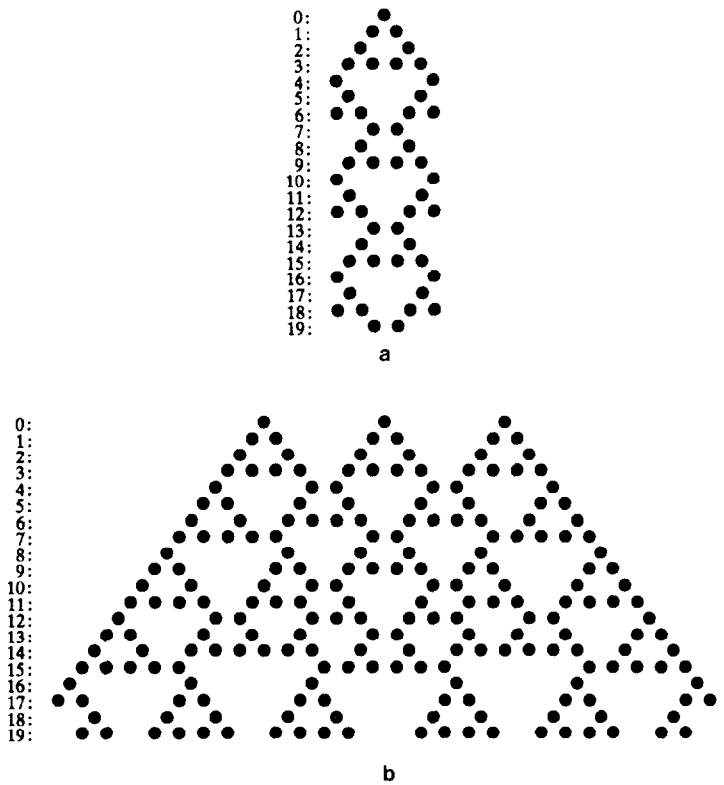


FIG. 4. (a) CA with null boundary conditions evolving from single nonzero seed, and (b) replacement of this CA by unbounded case accomplished by introducing two image charges. Note that the two are identical from 0 to L, where $L=8$ here. For a sufficiently large number of time steps $T \gtrsim L$ additional (second order) images must be introduced.

COMPUTATION VIA IMAGE CHARGES VERSUS DIRECT SIMULATION

The normal procedure which is followed in arriving at the time series of CA configurations given a particular initial state is direct simulation. On each time step the successive state of each node is computed from the previous state of its neighbours. For a one-dimensional CA of length L and for T time steps, this involves LT computations. This computational effort is required even if one is only interested in a few, or perhaps a single, site value.

By the method of image charges described above, the computational effort can often be greatly reduced. For example, in arriving at the value of a given site after T time steps which evolves from an initial configuration with R nonzero sites, Eq. (1) represents a single computation involving a sum over R terms. However, for the bounded CA we must add to this the contribution from the image charges. The number of these charges required will depend upon T, since successively more

images must be introduced for longer times as discussed above. The number of image charges required per nonzero site is given by $2T/L$ for $T \gg L$. This results in a number of computational steps per site value upper bounded by $3RT/L$ at time T. Direct simulation requires LT computational steps so the method of image charges is faster by a factor $L^2/3R$.

In computing the entire (global) configuration all L sites must be evaluated at time T, so the improvement factor reduces to $L/3R$. This is significant provided the number of nonzero sites $R \ll L$. It happens that with the CA rule under consideration, regardless of the value of R in the original ($T=0$) configuration, running a direct simulation for a relatively small number of time steps will always result in a configuration with $R \ll L$ (see Fig. 2a, e.g., at $T=4, 8, 16,...$). Therefore the computational effort for arriving at the global configuration (all site values) may be dramatically reduced, in the event of a large initial R, by evolving a few configurations by direct simulation until a configuration with a sufficiently small R is obtained, and then applying the image method to this configuration to obtain the results for later time steps. Since R is of order unity for the "intermediate" configuration and this intermediate configuration may be arrived at in a number of time steps which is also of order unity, the improvement factor of the image method for the entire configuration, over direct simulation, is a factor approaching L for $L \gg 1$ and $T \gg L$.

Therefore, the present method of image charges results in a reduced computational effort by a factor of order L^2 for a single site value, or L for the entire CA configuration, for sufficiently large time T.

As a final point offered by a reviewer, it should be recognized that the image charge approach is not unique to null boundary conditions. For periodic boundaries, the (finite) initial state is equivalent to an infinite lattice with periodic repetition of this state (the images of the initial state).

Conclusions

Cellular automata provide a class of VLSI building blocks which effectively exploit the desirable features of local communications requirements and simple (small area) repetitive logical nodes. The symmetrical geometry dramatically reduces design time and enhances testability and reliability. The computational effort involved in the analysis of the evolution of CA configurations, for the particular case of linear rules and bounded CA (those having null boundary conditions natural for VLSI implementation) is dramatically reduced using the present method of image charges. Extensions of this method to other linear rules, apart from the simple mod 2 addition of the two neighbour sites presented here, and to two-dimensional CA are expected to be fairly straightforward. With appropriate analytical methods such as those advanced in this paper, the CA may find many applications in the design of digital VLSI systems, for example, as ALU registers or as finite state machines in microprogram controllers.

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